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Stationary spherically symmetric one-kink metric in general relativity

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Abstract. It can be shown that general relativity admits structures that are strictly conserved in number for topological reasons. Such objects, called 'kinks', are associated with the particular form of the metric and it has been shown that a kink has many of the properties expected of a classical analogue of a Fermi particle. This paper examines the general form of a stationary spherically symmetric metric corresponding to a single kink centred at the origin. This is a generalization of the usual spherically symmetric Schwarzschild case. The Einstein tensor is deduced for this system, and it is shown that there is no free-field solution for this case. It is indicated how possible extensions, such as the inclusion of electric charge, may lead to the possibility of solutions.

1. Introduction

The 'theory of kinks' (Finkelstein and Misner 1959, Skyrme 1958, 1961a, b) is a theory of a particular type of particle-like solution that can occur in certain nonlinear (classical) field theories. The theory has been largely developed with a view to interpreting a 'kink' as an elementary particle†. To every kink there will be an anti-kink, and, by virtue of the topological configuration of the field variables and the boundary conditions at infinity, the algebraic number of kinks will be conserved. This can be seen to be true even after quantization (Williams 1972). Furthermore, the possibility of interchanging the positions of two kinks leads to the usual field-theoretic idea of odd or even statistics. For field theories in three-dimensional space, it is possible to define the concept of 'spin', both even- and odd-half-integer, *even at the classical level.* This has been done by Finkelstein and Misner (Finkelstein and Misner 1959, Finkelstein 1966) using topological ideas of connectedness. Once more, a spin-statistics theorem, of the type usual in linear relativistic quantum field theories, has been proved for kink theories by Finkelstein and Rubinstein (1968). Thus a classical field theory containing kinks of odd-half-integer spin (in the sense of the definition of Finkelstein and Misner) provides the classical analogue of a quantum field theory of fermions. Much of the importance of kink theory lies in this ability to describe Fermi particles, since the latter play a central role in elementary particle physics.

In this paper we shall choose general relativity as the classical field theory and shall study the properties of the metric $g_{\mu\nu}$ when there is a kink present. First, however, by way of illustration, we shall examine some simpler field theories. A one-dimensional

† For other physical interpretations, see especially Enz (1964), Barone *et al* (1971), and Caudrey *et al* (1973).

example is provided by considering the set of all mappings θ from the real line R^1 into the circle S^1,

$$\theta : R^1 \rightarrow S^1.$$

Heuristically, the number of kinks present is equal to the number of times that the real line wraps itself around the circle (ie the Brouwer degree of the mapping). Thus a single kink is pictured as a twist through 2π. Since a mapping is by definition continuous, it follows that the twist must occur over a finite region of space so that by its very nature a kink is an extended object rather than a point particle. It is important to specify boundary conditions at infinity. Let us denote any member of R^1 by x, and parametrize S^1 by two real variables (ϕ_1, ϕ_2) whose squares add up to unity. Then the boundary conditions are chosen to be

$$\theta(x) \rightarrow (0, 1) \qquad \text{as } x \rightarrow \pm\infty.$$

This is like saying that there is no 'matter' at infinity. Such boundary conditions imply that the number of kinks in the system is conserved for all time, independent of any particular Lagrangian that may be chosen for the system. This model has been extensively studied (Barone *et al* 1971, Caudrey *et al* 1973, Enz 1963, 1964, Perring and Skyrme 1962, Rubinstein 1970, Scott 1969, 1970, Streater and Wilde 1970). In particular, Skyrme (1961a) has quantized the model and has explicitly constructed the quantum-mechanical operator that creates a kink. Such operators are seen to satisfy *anti-commutation* relations.

This simple model suggests a generalization to three dimensions. Thus consider mappings φ from three-dimensional space R^3 into the three-sphere S^3,

$$\varphi : R^3 \rightarrow S^3.$$

Parametrizing S^3 by four real variables $(\phi_1, \phi_2, \phi_3, \phi_4)$ such that

$$\phi_1^2 + \phi_2^2 + \phi_3^2 + \phi_4^2 = 1$$

we ensure the conservation of kinks by imposing the boundary conditions

$$\varphi(\boldsymbol{x}) \rightarrow (0, 0, 0, 1) \qquad \text{as } |\boldsymbol{x}| \rightarrow \infty$$

where $\boldsymbol{x}$ is any member of R^3. Using topological arguments it can be shown (Williams 1970) that wavefunctionals in this model will exhibit a double-valuedness characteristic of spin $\frac{1}{2}$. Both classical and quantum-mechanical versions of this model have been studied by Skyrme (1961b, 1962, 1971) who, by introducing a suitable dynamics for the boson fields $\{\phi_\mu\}$, shows that the kink field will satisfy the Dirac equation, in the lowest approximation. The model also bears simularities with chiral dynamics, and has been discussed from this viewpoint by Dowker (1972a, b).

We now turn to what is the principal purpose of this paper, the examination of general relativity from the point of view of kink theory. Here, one is concerned with a metric tensor $g_{\mu\nu}$ which belongs to the set $S_{4,1}$ of all 4×4 real symmetric matrices of signature $(+\ +\ +\ -)$. For the majority of this paper we shall regard space–time as simply connected, and so the usual geons of geometrodynamics such as Einstein–Rosen bridges and wormholes will not be considered. Thus we can use a space coordinate $\boldsymbol{x} \in R^3$, and at any particular instant of time, $g_{\mu\nu}$ can be regarded as a mapping from R^3 into $S_{4,1}$:

$$g_{\mu\nu} : R^3 \rightarrow S_{4,1}.$$

We impose the boundary condition that

$$g_{\mu\nu} \to \eta_{\mu\nu} \qquad \text{as } |\boldsymbol{x}| \to \infty$$

where $\eta_{\mu\nu}$ is the Minkowski metric, diag$\{1, 1, 1, -1\}$. The occurrence of kinks in general relativity was first discussed by Finkelstein and Misner (1959). They showed that kinks arise naturally within the theory and that they are of a single type, so that the kink number is a single integer. Since $g_{\mu\nu}$ is a mapping between spaces of different dimensions, the concept of Brouwer degree cannot be used to count the number of kinks. However, the different possibilities can be analysed by considering the third homotopy group of $S_{4,1}$, namely $\pi_3(S_{4,1})$. It turns out (Steenrod 1951) that $\pi_3(S_{4,1})$ is isomorphic with the group of integers,

$$\pi_3(S_{4,1}) = Z.$$

The group element of $\pi_3(S_{4,1})$ corresponding to the zero element of Z will contain all of the metrics $g_{\mu\nu}$ corresponding to zero kink number. The one-kink metrics will be contained in the element which generates $\pi_3(S_{4,1})$ (and the one-anti-kink metrics will be contained in the inverse of this element). To investigate the possibility that the kinks of general relativity have odd-half-integer spin one must examine 2π rotation paths in that part of mapping space which contains the one-kink mappings. This, it has been shown by Finkelstein and Misner (1959), is equivalent to investigating the structure of $\pi_4(S_{4,1})$. The fact that

$$\pi_4(S_{4,1}) = Z_2$$

where Z_2 is the group of integers *modulo* 2, means that there will be non-trivial, doubly-connected paths, and it has been shown (Williams 1971) that such paths arise from rotation through 2π. Thus the kinks of general relativity are objects with odd-half-integer spin. If one adopts the philosophy that elementary particle physics can be described in terms of the curvature of space, then solutions of Einstein's equations corresponding to a one-kink metric would provide classical analogues for the fermions.

In what follows, we construct a general form of the metric for a single kink, centred at the origin, with the simplifying restrictions that the system is spherically symmetric and stationary in time. This is a generalization of a simple model for a one-kink metric previously considered by Williams and Zia (1973). We then calculate the Einstein tensor G_μ^ν. As it has been pointed out by Finkelstein and Misner (1959, p 239), if there are kinks present it is not clear whether or not the free-space Einstein equations possess any solutions. To study this question we investigate whether or not our one-kink metric can lead to such a solution. Finally, we discuss the possibility that the kink may be electrically charged.

2. Construction of the metric

$S_{4,1}$ is a fibre bundle with the three-dimensional rotation group SO_3 as base. Because of the boundary conditions, the infinite boundary of R^3 can be identified to a single point resulting in a manifold topologically equivalent to S^3. It can be shown (Steenrod 1951) that an example of a one-kink mapping is one which maps the whole of R^3 ($\simeq S^3 \simeq SU_2$) onto the SO_3 base using the usual two–one homomorphism between the groups S^3 and SO_3. Any mapping homotopic to (ie deformable into) this mapping will also be a one-kink mapping. It is not difficult to see that an example of such a mapping is provided by the metric

$$\|g_{\mu\nu}\| = P\begin{pmatrix} e^{\sigma} & & & \\ & e^{\sigma} & & \\ & & e^{\sigma} & \\ & & & -e^{\lambda} \end{pmatrix}P^{-1}$$

where P is an orthogonal matrix given by

$$P = \begin{pmatrix} -\phi_4 & -\phi_3 & \phi_2 & -\phi_1 \\ \phi_3 & -\phi_4 & -\phi_1 & -\phi_2 \\ -\phi_2 & \phi_1 & -\phi_4 & -\phi_3 \\ \phi_1 & \phi_2 & \phi_3 & -\phi_4 \end{pmatrix}.$$

The $(\phi_1, \phi_2, \phi_3, \phi_4) \in S^3$, and are functions of $\boldsymbol{x}$ representing a one-kink mapping from R^3 onto S^3 (and so cannot be deformed away). In actual fact, SO_3 is involved rather than S^3, since giving a value to $g_{\mu\nu}$ only determines the $\{\phi_\mu\}$ to within a $\pm$ sign. Thus the metric $g_{\mu\nu}$ is topologically equivalent to a mapping from R^3 onto the base of $S_{4,1}$ and hence is a one-kink mapping. The functions σ, λ are assumed to be real valued functions of $r = |\boldsymbol{x}|$. We choose the particular form of the $\{\phi_\mu\}$ to be

$$\phi_i = \frac{x^i}{r}\sin\alpha \qquad i = 1, 2, 3$$

$$\phi_4 = \cos\alpha$$

where α is any real function of r such that

$$\alpha(0) = \pi; \qquad \alpha(\infty) = 0. \tag{1}$$

This ensures the presence of a kink. To obtain asymptotic flatness we also insist on the boundary conditions

$$\sigma(r), \lambda(r) \to 0 \qquad \text{as } r \to \infty. \tag{2}$$

We note that the above metric is a combination of the simple model given by Williams and Zia (1973) with the usual spherically symmetric case that leads to the Schwarzschild solution. It will be convenient to exploit spherical symmetry to the utmost and so we write the metric in spherical polar coordinates. We allow Greek labels to run over r, θ, φ, t. Hence we define the one-kink metric $g_{\mu\nu}$ to be given by:

$$g_{rr} = e^{\sigma} - (e^{\sigma} + e^{\lambda})\sin^2\alpha$$

$$g_{\theta\theta} = r^2$$

$$g_{\varphi\varphi} = r^2\sin^2\theta$$

$$g_{tt} = e^{\sigma}\sin^2\alpha - e^{\lambda}\cos^2\alpha$$

$$g_{rt} = g_{tr} = -(e^{\sigma} + e^{\lambda})\sin\alpha\cos\alpha$$

all other components being zero. Here we have used the usual trick of changing the scale to remove the factor of e^{σ} in front of the $\theta\theta$ and $\varphi\varphi$ components. If there were no kinks present (ie $\alpha \equiv 0$) the $g_{\mu\nu}$ reduce to the usual spherically symmetric form as given by Tolman (1934), for example. The special case of $\sigma = \lambda = 0$ leads back to the simple model mentioned previously (Williams and Zia 1973). An essential feature of our form for $g_{\mu\nu}$, and indeed of all metrics that contain a kink, is that g_{rt} is not identically zero. It is important that g_{tt} take both positive and negative values. Thus g_{tt} will be zero at least once. It is this which prevents our using the usual coordinate transformation to transform away the g_{rt} term. To illustrate this point, we follow the procedure of Bergmann (1942). Writing the general spherically symmetric line element in the form

$$ds^2 = A(r)\,dt^2 + 2B(r)\,dr\,dt + C(r)\,dr^2 + D(r)\,d\theta^2 + E(r)\,d\varphi^2$$

a coordinate transformation is introduced by

$$\bar{t} = t + f(r)$$
$$\bar{r} = r, \qquad \bar{\theta} = \theta, \qquad \bar{\varphi} = \varphi.$$

The components g_{rt} transform according to

$$\bar{g}_{rt} = g_{rt} + g_{tt}\frac{\partial t}{\partial \bar{r}}$$

or

$$\bar{B} = B - A\frac{df}{dr}.$$

One then tries to eliminate the $\bar{B}$ term by choosing f so that it satisfies the equation

$$\frac{df}{dr} = \frac{B}{A}.$$

However, in our case

$$A = g_{tt} = e^{\sigma}\sin^2\alpha - e^{\lambda}\cos^2\alpha$$

and since there is a kink present, A will take the value zero at least somewhere, thereby invalidating the above procedure. We note in passing, that although deformations of the above type cannot remove the kink, given *any* stationary spherically symmetric one-kink metric we can use similar kinds of deformations to put the metric into the form given, that is with the additional variable occurring in combinations $\sin\alpha$ and $\cos\alpha$. This will lead to considerable simplification.

Using our form for the metric, $g_{\mu\nu}$, we now proceed to calculate the G_{μ}^{ν}. Even though there is doubt as to the role that such kink metrics may play in physics, we can still take the view that the above metric is worthy of investigation in its own right, its being a non-trivial generalization of the usual spherically symmetric case.

3. Calculation of the Einstein tensor

The inclusion of the additional variable $\alpha(r)$ in the $g_{\mu\nu}$ greatly complicates the algebra involved. The latter may be performed more conveniently, however, by introducing the following variables:

$$P = \sin^3\alpha \cos\alpha$$
$$Q = \sin\alpha \cos^3\alpha$$
$$\Lambda_+ = 1 + e^{\sigma-\lambda}$$
$$\Lambda_- = 1 + e^{-\sigma+\lambda}$$
$$S_+ = 1 - P\Lambda_+/(P+Q)$$
$$T_+ = 1 - Q\Lambda_+/(P+Q).$$

Using a prime to denote the derivative with respect to r, we write

$$\xi = \sigma'/2, \qquad \eta = \lambda'/2.$$

Much use will also be made of the relations

$$P+Q = \sin\alpha\cos\alpha$$
$$PQ = (P+Q)^4$$
$$\frac{P}{P+Q} = \sin^2\alpha$$
$$\frac{Q}{P+Q} = \cos^2\alpha.$$

It is now a simple matter to calculate the Christoffel symbols according to

$$\Gamma^{\gamma}_{\mu\nu} = \tfrac{1}{2}g^{\gamma\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^{\nu}} + \frac{\partial g_{\beta\nu}}{\partial x^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial x^{\beta}}\right).$$

These are given in the appendix. We note that

$$g \equiv \det g_{\mu\nu} = -r^4 \sin^2\theta\, e^{\sigma+\lambda}$$
$$\Gamma^{\beta}_{r\beta} = \frac{2}{r} + \xi + \eta. \tag{3}$$

The presence of the kink has no effect on either of these two expressions. The Ricci tensor

$$R_{\mu\nu} = \frac{\partial\Gamma^{\gamma}_{\mu\nu}}{\partial x^{\gamma}} - \frac{\partial\Gamma^{\beta}_{\mu\beta}}{\partial x^{\nu}} + \Gamma^{\gamma}_{\mu\nu}\Gamma^{\beta}_{\gamma\beta} - \Gamma^{\gamma}_{\mu\beta}\Gamma^{\beta}_{\nu\gamma}$$

can be found in a straightforward manner and is listed for reference in the appendix, along with the curvature scalar R. The Einstein tensor is given by

$$G^{\nu}_{\mu} = R^{\nu}_{\mu} - \tfrac{1}{2}\delta^{\nu}_{\mu}R$$

and the non-trivial components are found to be:

$$G^{r}_{r} = \frac{e^{-\sigma}}{r}\left[\alpha'[-2(P+Q)\Lambda_+] + \xi\left(-\frac{2P}{P+Q}(\Lambda_+ - 1)\right) + \eta\left(+\frac{2Q}{P+Q}\right) + \left(\frac{S_+ - e^{\sigma}}{r}\right)\right]$$

$$G_\theta^\theta = G_\varphi^\varphi = e^{-\sigma}\Bigg[\left(\alpha''+\frac{2}{r}\alpha'\right)[-(P+Q)\Lambda_+]+\xi'\left(-\frac{P}{P+Q}(\Lambda_+-1)\right)+\eta'\left(+\frac{Q}{P+Q}\right)$$

$$+\frac{\xi}{r}\left(\frac{P-Q}{P+Q}-\frac{P}{P+Q}\Lambda_+\right)+\frac{\eta}{r}\left[-\left(\frac{P-Q}{P+Q}\right)+\frac{P}{P+Q}\Lambda_+\right]$$

$$+(\alpha')^2\left(\frac{P^2-Q^2}{(P+Q)^2}\Lambda_+\right)+(\xi)^2\left(-\frac{P}{P+Q}(\Lambda_+-1)\right)+(\eta)^2\left(\frac{Q}{P+Q}\right)$$

$$+(\alpha'\xi)[4(P+Q)-3(P+Q)\Lambda_+]+(\alpha'\eta)[-4(P+Q)+(P+Q)\Lambda_+]$$

$$+(\xi\eta)\left(-1+\frac{P}{P+Q}\Lambda_+\right)\Bigg]$$

$$G_t^t = \frac{e^{-\sigma}}{r}\left[\alpha'[-2(P+Q)\Lambda_+]+\xi\left(-\frac{2Q}{P+Q}\right)+\eta\left(\frac{2P}{P+Q}(\Lambda_+-1)\right)+\left(\frac{S_+-e^{\sigma}}{r}\right)\right]$$

$$G_t^r = 0$$

$$G_r^t = \frac{-2\Lambda_+ e^{-\sigma}}{r}(P+Q)(\xi+\eta).$$

Because there is a kink present, equation (1) implies

$$P+Q \not\equiv 0. \tag{4}$$

For this reason, G_r^t is non-trivial.

4. Attempt to find a free-field solution

Let us consider the Einstein equations,

$$G_\mu^\nu = 8\pi G T_\mu^\nu \tag{5}$$

where G is the gravitational constant and T_μ^ν is the energy–momentum tensor for the system. In this paper we shall follow the approach of Misner and Wheeler (1957) in regarding classical physics as describable in terms of curved empty space, and nothing more. Thus T_μ^ν will be identically zero unless there is an electromagnetic field present. In the latter case, T_μ^ν will be simply the electromagnetic energy–momentum tensor. (The electromagnetic field is also described in terms of the curvature of space, being given by the 'Maxwell square root' of the Ricci tensor; we refer to Misner and Wheeler (1957).)

We mention in passing, a different approach that is adopted in the paper by Ross (1972) in which an elementary particle is studied from a gravitational point of view by using the Reissner–Nordstrøm metric (no kinks). Ross regards the strong interaction of elementary particle physics not as a particular species of gravitational force but rather as a separate force in its own right, and so he includes it explicitly in the energy–momentum tensor on the right-hand side of equation (5). The procedure is of necessity phenomenological, nevertheless, by choosing a Yukawa potential for T_t^t, he obtains some interesting results.

Returning to the problem in hand, we now consider the situation in which we have an uncharged kink. Thus we neglect electromagnetic forces and look for solutions of the free field equations, $G^{\nu}_{\mu} = 0$. Bearing equation (4) in mind, $G^{t}_{r} = 0$ implies

$$\xi + \eta = 0.$$

Using the boundary conditions of equation (2), this gives

$$\lambda = -\sigma.$$

We now make this replacement in all of the G^{ν}_{μ}. We use $\tilde{G}^{\nu}_{\mu}$, $\tilde{S}_{+}$, $\tilde{\Lambda}_{+}$ to denote G^{ν}_{μ}, S_{+}, Λ_{+} with λ put equal to $-\sigma$:

$$\tilde{G}^{r}_{r} = \tilde{G}^{t}_{t} = \frac{e^{-\sigma}}{r}\left[\alpha'[-2(P+Q)\tilde{\Lambda}_{+}] + \xi\left(\frac{2(P-Q)}{P+Q} - \frac{2P}{P+Q}\tilde{\Lambda}_{+}\right) + \left(\frac{\tilde{S}_{+} - e^{\sigma}}{r}\right)\right] \tag{6}$$

$$\begin{aligned}\tilde{G}^{\theta}_{\theta} = \tilde{G}^{\varphi}_{\varphi} = e^{-\sigma}\Bigg[&\left(\alpha'' + \frac{2}{r}\alpha'\right)[-(P+Q)\tilde{\Lambda}_{+}] + \xi'\left(\frac{P-Q}{P+Q} - \frac{P}{P+Q}\tilde{\Lambda}_{+}\right)\\ &+ \frac{\xi}{r}\left(\frac{2(P-Q)}{P+Q} - \frac{2P}{P+Q}\tilde{\Lambda}_{+}\right) + (\alpha')^{2}\left(\frac{P-Q}{P+Q}\tilde{\Lambda}_{+}\right)\\ &+ (\xi)^{2}\left(2 - \frac{2P}{P+Q}\tilde{\Lambda}_{+}\right) + (\alpha'\xi)[8(P+Q) - 4(P+Q)\tilde{\Lambda}_{+}]\Bigg].\end{aligned} \tag{7}$$

Thus there will be two equations for the two unknowns, σ and α. Applying the first equation, $\tilde{G}^{r}_{r} = 0$, we may eliminate α' and ξ/r terms from $\tilde{G}^{\theta}_{\theta}$ in equation (7). We may also eliminate the α'' and ξ' terms from $\tilde{G}^{\theta}_{\theta}$ by applying $\partial_{r}\tilde{G}^{r}_{r} = 0$. Putting $\tilde{G}^{\theta}_{\theta} = 0$ then gives

$$\tfrac{1}{2}\partial_{r}\left(\frac{\tilde{S}_{+} - e^{\sigma}}{r}\right) + \frac{1}{r}\left(\frac{\tilde{S}_{+} - e^{\sigma}}{r}\right) = \frac{(\sigma')^{2}}{2}\left[-\left(\frac{P-Q}{P+Q}\right) + \frac{P}{P+Q}\tilde{\Lambda}_{+}\right]$$

which, after some cancellation, leads to

$$\sigma'\left(\frac{Q}{P+Q} + \frac{P}{P+Q}e^{2\sigma}\right) = \frac{1}{r}\left(\frac{Q}{P+Q} - \frac{P}{P+Q}e^{2\sigma} - e^{\sigma}\right).$$

This can be rewritten as

$$\xi\left(\frac{2(P-Q)}{P+Q} - \frac{2P}{P+Q}\tilde{\Lambda}_{+}\right) + \left(\frac{\tilde{S}_{+} - e^{\sigma}}{r}\right) = 0$$

and is one of the terms in the expression for $\tilde{G}^{r}_{r}$, equation (6). The equation $\tilde{G}^{r}_{r} = 0$ then becomes

$$\frac{e^{-\sigma}}{r}\alpha'(P+Q)(1 + e^{2\sigma}) = 0.$$

Clearly, since a kink is present, this equation has no solution (see equation (4)). Thus we conclude that there are no stationary spherically symmetric kink solutions to the free field equations.

5. Electrically charged kink

We denote the covariant electromagnetic field tensor by $F_{\mu\nu}$. For the present spherically symmetric time independent problem, there will be an electrostatic field characterized by a scalar potential ϕ, and the only non-zero components of $F_{\mu\nu}$ will be given by

$$F_{tr} = -F_{rt} = \partial_r \phi.$$

The contravariant components are

$$F^{tr} = -e^{-\sigma-\lambda}F_{tr}.$$

The electromagnetic field equation is

$$\partial_r F^{tr} + F^{tr}\Gamma^{\beta}_{r\beta} = 0.$$

As seen from equation (3), even with the kink present, $\Gamma^{\beta}_{r\beta}$ takes on the same value as in the normal Schwarzschild case. Thus the electromagnetic equation will have the usual form of first integral:

$$r^2 e^{-\frac{1}{2}(\sigma+\lambda)}\partial_r \phi = -e$$

where e is a constant of integration representing the electric charge. Writing T^{ν}_{μ} as the energy–momentum tensor of the electromagnetic field, we have

$$T^{\nu}_{\mu} = \frac{1}{4\pi}(F_{\mu\gamma}F^{\nu\gamma} - \tfrac{1}{4}\delta^{\nu}_{\mu}F_{\gamma\beta}F^{\gamma\beta}).$$

Einstein's equations then become

$$G^r_r = \frac{Ge^2}{r^4}$$

$$G^t_t = \frac{Ge^2}{r^4}$$

$$G^{\nu}_{\mu} = 0 \qquad \text{for all other } \mu, \nu.$$

Since the equation $G^t_r = 0$ still holds, it follows that $\lambda = -\sigma$ and so the above equations reduce to

$$\tilde{G}^r_r = \frac{Ge^2}{r^4}$$

$$\tilde{G}^\theta_\theta = 0$$

with the $\tilde{G}^r_r$ and $\tilde{G}^\theta_\theta$ given by equations (6) and (7). Following a procedure similar to that taken for the uncharged case, we arrive at

$$(\sigma'\alpha')[-2(P+Q)(1+e^{2\sigma})] = \frac{Ge^2}{r^4}(1+e^{\sigma}).$$

Clearly, with charge present, the possibilities of finding solutions are much wider. The above approach, however, is more suitable for a system with a point charge. This is not very appropriate for a kink since the latter has spatial extension. For this reason, we end the analysis at this point.

6. Conclusions

In this paper we have constructed the metric appropriate to a single stationary spherically symmetric kink placed at the origin. It was shown that there are no free-field solutions of this type. To the extent that such gravitational kinks represent a serious model of an elementary particle, such a free-field solution should correspond to a neutral particle whose behaviour is time independent (and so must be stable). However, there is no suitable candidate in the table of known elementary particles. (The only neutral stable particle is the neutrino, but this is ineligible because its velocity is that of light so that it cannot be regarded as fixed at the origin.) Thus the lack of a free-field solution is quite a satisfactory situation.

It was pointed out that the introduction of electrical charge could lead to more interesting possibilities. Here one might hope to construct a classical model for the stable charged fermions that occur in nature, namely the proton and electron, and their antiparticles. However, a kink is an extended object, and it would seem suitable to introduce an electric charge of similar spatial nature. A way of doing this is provided by the geons of Wheeler's geometrodynamics (Misner and Wheeler 1957, Wheeler 1962). Some situations which can arise when a kink is combined with an Einstein–Rosen bridge have already been discussed by Finkelstein and Misner (1962).

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Appendix. Christoffel symbols and Ricci tensor

$$\Gamma^r_{rr} = \alpha'(-P\Lambda_- + Q\Lambda_+) + \xi\left(\frac{Q}{P+Q} + (P+Q)^2\Lambda_+\right) + \eta\left(\frac{P}{P+Q} + (P+Q)^2\Lambda_-\right)$$

$$\Gamma^r_{\theta\theta} = -rS_+ \,\mathrm{e}^{-\sigma}$$

$$\Gamma^r_{\varphi\varphi} = \sin^2\theta\, \Gamma^r_{\theta\theta}$$

$$\Gamma^r_{tt} = \alpha'(P\Lambda_+ - Q\Lambda_-) + \xi\left(-\frac{P}{P+Q}S_+\right) + \eta\left(\frac{Q}{P+Q}(\Lambda_- - 1)S_+\right)$$

$$\Gamma^r_{rt} = -\Gamma^t_{tt} = \alpha'[-(P+Q)^2(\Lambda_+ + \Lambda_-)] + \xi(-P\Lambda_+) + \eta(Q\Lambda_-)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\varphi_{r\varphi} = 1/r$$

$$\Gamma^t_{rt} = \alpha'(P\Lambda_- - Q\Lambda_+) + \xi\left(\frac{P}{P+Q}T_+\right) + \eta\left(-\frac{Q}{P+Q}(\Lambda_- - 1)T_+\right)$$

$$\Gamma^t_{rr} = \alpha'\left(\frac{P}{P+Q}\Lambda_- + \frac{Q}{P+Q}\Lambda_+ - (P+Q)^2(\Lambda_+ + \Lambda_-)\right) + \xi[Q\Lambda_+ - 2(P+Q)]$$
$$+ \eta(-P\Lambda_- + 2(P+Q)]$$

$$\Gamma^t_{\theta\theta} = r(P+Q)\Lambda_+ \,\mathrm{e}^{-\sigma}$$

$$\Gamma^t_{\varphi\varphi} = \sin^2\theta\, \Gamma^t_{\theta\theta}$$

$$\Gamma^\varphi_{\theta\varphi} = \cot\theta$$

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$

$$R_{rr} = -(\xi' + \eta') + \left(\alpha'' + \frac{2}{r}\alpha'\right)(-P\Lambda_- + Q\Lambda_+)$$
$$+ \left(\xi' + \frac{2}{r}\xi\right)\left(\frac{Q}{P+Q} + (P+Q)^2\Lambda_+\right) + \left(\eta' + \frac{2}{r}\eta\right)\left(\frac{P}{P+Q} + (P+Q)^2\Lambda_-\right)$$
$$+ (\alpha')^2\left(-2(P+Q)^2(\Lambda_+ + \Lambda_-) + \frac{P}{P+Q}\Lambda_- + \frac{Q}{P+Q}\Lambda_+\right)$$
$$+ (\xi)^2\left(-\frac{P^2}{(P+Q)^2} + (P+Q)^2(\Lambda_+ - 1)\right)$$
$$+ (\eta)^2\left(-\frac{Q^2}{(P+Q)^2} + (P+Q)^2(\Lambda_- - 1)\right)$$
$$+ (\alpha'\xi)[-4(P+Q) + P\Lambda_- + 3Q\Lambda_+] + (\alpha'\eta)[4(P+Q) - 3P\Lambda_- - Q\Lambda_+]$$
$$+ (\xi\eta)[1 - (P+Q)^2(\Lambda_+ + \Lambda_-)]$$

$$R_{\theta\theta} = (1 - S_+\,\mathrm{e}^{-\sigma}) + r\,\mathrm{e}^{-\sigma}\left[\alpha'[2(P+Q)\Lambda_+]\right.$$
$$\left. + \xi\left(1 - \frac{2P}{P+Q} + \frac{P}{P+Q}\Lambda_+\right) + \eta\left(-1 + \frac{2P}{P+Q} - \frac{P}{P+Q}\Lambda_+\right)\right]$$

$$R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$$

$$R_{tt} = \left(\alpha'' + \frac{2}{r}\alpha'\right)(P\Lambda_+ - Q\Lambda_-) + \left(\xi' + \frac{2}{r}\xi\right)\left(-\frac{P}{P+Q}S_+\right)$$
$$+ \left(\eta' + \frac{2}{r}\eta\right)\left(\frac{Q}{P+Q}(\Lambda_- - 1)S_+\right)$$
$$+ (\alpha')^2\left(2(P+Q)^2(\Lambda_+ + \Lambda_-) - \frac{P}{P+Q}\Lambda_+ - \frac{Q}{P+Q}\Lambda_-\right)$$
$$+ (\xi)^2\left(-\frac{P}{P+Q}S_+\right) + (\eta)^2\left(\frac{Q}{P+Q}(\Lambda_- - 1)S_+\right)$$

$$+(\alpha'\xi)[-4(P+Q)+3P\Lambda_++Q\Lambda_-]+(\alpha'\eta)[4(P+Q)-P\Lambda_+-3Q\Lambda_-]$$

$$+(\xi\eta)\left(1-\frac{P^2}{(P+Q)^2}\Lambda_+-\frac{Q^2}{(P+Q)^2}\Lambda_-\right)$$

$$R_{rt}=R_{tr}=\left(\alpha''+\frac{2}{r}\alpha'\right)[-(P+Q)^2(\Lambda_++\Lambda_-)]$$

$$+\left(\xi'+\frac{2}{r}\xi\right)(-P\Lambda_+)+\left(\eta'+\frac{2}{r}\eta\right)(Q\Lambda_-)+(\alpha')^2[(P-Q)(\Lambda_++\Lambda_-)]$$

$$+(\xi)^2(-P\Lambda_+)+(\eta)^2(Q\Lambda_-)+(\alpha'\xi)[-3(P+Q)^2\Lambda_++(P+Q)^2\Lambda_-]$$

$$+(\alpha'\eta)[(P+Q)^2\Lambda_+-3(P+Q)^2\Lambda_-]+(\xi\eta)(P\Lambda_+-Q\Lambda_-)$$

$$R=\mathrm{e}^{-\sigma}\Bigg[-\frac{2}{r}\left(\frac{S_+-\mathrm{e}^{\sigma}}{r}\right)+\alpha''[2(P+Q)\Lambda_+]+\frac{\alpha'}{r}[8(P+Q)\Lambda_+]$$

$$+\xi'\left(\frac{2P}{P+Q}(\Lambda_+-1)\right)+\frac{\xi}{r}\left(\frac{4Q}{P+Q}+\frac{4P}{P+Q}(\Lambda_+-1)\right)$$

$$+\eta'\left(-\frac{2Q}{P+Q}\right)+\frac{\eta}{r}\left(-\frac{4Q}{P+Q}-\frac{4P}{P+Q}(\Lambda_+-1)\right)$$

$$+(\alpha')^2\left(\frac{2(-P+Q)}{P+Q}\Lambda_+\right)+(\xi)^2\left(\frac{2P}{P+Q}(\Lambda_+-1)\right)+(\eta)^2\left(-\frac{2Q}{P+Q}\right)$$

$$+(\alpha'\xi)[-8(P+Q)+6(P+Q)\Lambda_+]+(\alpha'\eta)[8(P+Q)-2(P+Q)\Lambda_+]$$

$$+(\xi\eta)\left(2-\frac{2P}{P+Q}\Lambda_+\right)\Bigg].$$

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